Estimation of low frequency phase velocity dispersion in weak contrast periodic media

*Yu. Roganov (Tesseract Technologies Inc., Ukraine), A. Stovas (NTNU, Trondheim), V. Roganov (Glushkov Institute of Cybernetics, Ukraine)

SUMMARY

Formulas for calculating the approximation of the dispersion of vertical slowness and phase velocities of Floquet waves in a periodic horizontally layered medium with anisotropic layers are derived. The accuracy of the calculations is demonstrated on a three-layer periodic medium with orthorhombic layers with different azimuths of symmetry planes.
Introduction

This work is devoted to the development of a method for estimating the dispersion of the velocities of plane waves propagating in periodic horizontally layered media with anisotropic layers. This topic is not new - the study of the properties of waves propagating in periodic horizontally layered media has been the subject of many works, for example Rytov, 1956; Brekhovskikh, 1980; Sibiryakov et al., 1980; Molotkov, 1982; Norris, 1993; Wang and Rokhlin, 2002; Stovas et al., 2013.

The periodic medium at a frequency tending to zero is homogeneous, and its parameters are calculated by special averaging over the period of the layer parameters (Backus, 1962; Schoenberg and Muir, 1989). In this case, the waves propagate without dispersion.

For nonzero frequencies, the Floquet theorem is applicable (Floquet, 1883; Yakubovich and Starzhinskii, 1975), the consequence of which is the existence of stationary packets consisting of different types of plane waves propagating in the layers up and down. The propagation of these packets (Floquet waves) in the spectral region is described by a system of differential equations of the first order. The matrix of this system depends on the frequency, horizontal slowness, and layer parameters. It is proportional to the logarithm of propagator of the period (monodromy matrix). The eigenvalues of this effective system matrix are the vertical slowness of the Floquet waves. They contain the dispersion information. To estimate the eigenvalues of the logarithm of the propagator, it is expanded into BCH (Baker-Campbell-Hausdorff) series in frequency (Roganov and Stovas, 2012).

In this series, the first term is the system matrix of the homogeneous Backus medium, the second and third terms are responsible for the dispersion of waves at low frequencies. The third term represented by a homogeneous polynomial of degree 3.

In Roganov et al. (2019), an approximation of the eigenvalues of the effective system matrix is found for a binary periodic medium with orthorhombic layers. This approximation depends on the product of perturbations of the system matrices of the layers in the period and is determined by a homogeneous second-degree polynomial from perturbations of the layer parameters.

In this paper, the results of the previous article are generalized to a periodic horizontally layered medium with an arbitrary number of layers in a period and any type of anisotropy in the layers.

Method

Let us consider a horizontally layered medium with $N$ anisotropic layers in a period. The propagation of plane waves in each layer in the spectral region is described by the matrix equation

$$ \frac{db}{dz} = i\omega A_j b, \quad (1) $$

where $A_j$ is the system matrix of the $j$-th layer, $b = (u_1, u_2, u_j, \tau_{13}, \tau_{23}, \tau_{33})^T$ is the vector consisting of the components of the displacement velocities and the stress tensor, $i = \sqrt{-1}$, $\omega$ is the circular frequency,

$$ A_j = \begin{pmatrix} B & C_{33}^{-1} \\ D & B^T \end{pmatrix}, \quad B = C_{33}^{-1}(p_j C_{33} + p_2 C_{32}), \quad D = \sum_{m,n=1,2} p_m p_n (C_{mn} C_{33}^{-1} C_{3n} - C_{mn}), \quad + \rho_j I, $$

$p_1, p_2$ are horizontal components of the slowness vector, $I$ is the identity $3 \times 3$ matrix, $\rho_j$ is density and $C_{mn}[p,q] = c_{pq}^{(j)}$ is the matrix consisting of the components of the elastic tensor of the $j$-th layer.

Let us denote $z_j$ the thickness of the $j$-th layer, $H = z_1 + \ldots + z_N$ the thickness of the period and $\alpha_j = z_j / H$. The system matrix $\tilde{A}(\omega)$ of an infinite periodic horizontally layered medium is determined by the relation

$$ \tilde{A}(\omega) = \frac{1}{i\omega H} \log P(\omega, p_1, p_2), \quad (2) $$

where

$$ P(\omega, p_1, p_2) = \exp(i\omega z_{N} A_N) \ldots \exp(i\omega z_1 A_1). \quad (3) $$
is the matrix propagator of the layers $j=1,..,N$ and is called the monodromy matrix (Braga and Herrmann, 1992). The system matrix $\tilde{A}(\omega)$ at low frequency $\omega$ is determined by the first three terms of the BCH series (Roganov and Stovas, 2012)

$$\tilde{A}(\omega) = F_0 + \frac{i\omega H}{2} F_1 - \frac{\omega^2 H^2}{12} F_2 + o(\omega^3),$$

where

$$F_0 = \sum_{k=1}^{N} A_n A_k, \quad F_1 = \sum_{m,n=1 \atop m \neq n}^{N} \alpha_m \alpha_n [A_m, A_n], \quad [A_m, A_n] = A_m A_n - A_n A_m,$$

$$F_2 = \sum_{m,n=1 \atop m \neq n}^{N} \alpha_m^2 \alpha_n [A_m, [A_m, A_n]] + 2 \sum_{m,n,k=1 \atop m \neq n \neq k}^{N} \alpha_m \alpha_n \alpha_k [A_m, [A_n, A_k]] + [A_k, [A_n, A_m]].$$

We denote the eigenvalues $q_j^{(0)}$, $(j=1,..,6)$ of the matrix $A_B$, and $\varphi_j^{(0)}$, $\psi_j^{(0)}$ are the corresponding right and left eigenvectors, $k_j = \psi_j^{(0)} \varphi_j^{(0)}$, and apply the perturbation theory of the second order of accuracy in frequency $\omega$ to estimate the eigenvalues $q_j$ of the matrix $\tilde{A}(\omega)$ (Lankaster, 1969; Madelung, 1964). As a result, we get:

$$k_j q_j = k_j q_j^{(0)} - \frac{\omega^2 H^2}{12} \psi_j^{(0)} F_2 \varphi_j^{(0)} - \frac{\omega^2 H^2}{4} \sum_{m,n=1 \atop m \neq n}^{N} k_m^{-1} \psi_m^{(0)} F_m \varphi_m^{(0)} \psi_j^{(0)} \varphi_j^{(0)} + o(\omega^3).$$

In the future, we will assume that the periodic medium is weak contrast, i.e. the parameters $\rho_j$, $c_{mp, nj}^{(2)}$ of the layers differ little from the corresponding parameters $\rho_B$, $c_{mp, nj}^{(B)}$ of the Backus averaging. In this case, $A_j = A_B + \Delta A_j$, $\Delta A_j << A_j$ and $\sum_j \alpha_j \Delta A_j = 0$.

An increment, $\Delta q_j^2 = q_j^2(\omega) - q_j^{(0)}$, of the square of the vertical slowness of the $j$-th wave in weak contrast medium, from relation (5) can be simplified, if the right-hand side of this equality is approximated by polynomials up to the second order with respect to $\Delta A_j$:

$$\Delta q_j^2 = -\frac{\omega^2 H^2}{6} \sum_{p, q=1,..,N}^{p \neq q} \alpha_p \alpha_q \tau_{pq} \psi_j^{(0)} \Delta A_p G_j \Delta A_q \varphi_j^{(0)} + o(\omega^3, \Delta^2 A_j),$$

where $G_j = A_B - q_j^{(0)} 1$, $\tau_{pq} = \begin{cases} \alpha_p + \alpha_{p+1} + \ldots + \alpha_q, & \text{if } p \leq q, \\ 0, & \text{if } p > q. \end{cases}$

and $\tau_{pq} = \begin{cases} 3 \left( \tau_{1,p-1} - \tau_{p+1,N} \right), & \text{if } p = q, \\ 4 \tau_{1,p-1} - 2 \left( \tau_{1,p-1} + \tau_{q+1,N} \right), & \text{if } p > q. \end{cases}$

Since, $\Delta q_j^2 = -2 \Delta v_j / v_j^3$, from relation (6) follows the formula for the dispersion of the phase wave velocity of $j$-th wave:

$$\Delta v_j = \frac{\omega^2 H^2 q_j^{(0)} k_j}{12 \left( q_j^{(0)} + p_1^2 + p_2^2 \right)^{3/2}} \sum_{p, q=1,..,N}^{p \neq q} \alpha_p \alpha_q \tau_{pq} \psi_j^{(0)} \Delta A_p G_j \Delta A_q \varphi_j^{(0)} + o(\omega^3, \Delta^2 A_j).$$

For a two-layer periodic medium ($N = 2$), are right the relations $t_{11} = \alpha_2 (3 \alpha_2 - 2)$, $t_{12} = 1 - 3 \alpha_1 \alpha_2$, $t_{22} = \alpha_1 (3 \alpha_1 - 2)$, $\Delta A_2 = -\alpha_1 / \alpha_2 \Delta A_1$, $\alpha_2 = 1 - \alpha_1$, and a formula (8). This is consistent with the result of the article (Roganov, et. al., 2019).
\[
\Delta q_j = \frac{\omega^2 H^2 q_j^{(0)} k_j^{-1} \alpha_j^2}{6} \psi_j^{(0)} \Delta A_j G_j \Delta A_j \phi_j^{(0)} + o\left(\omega^2, \Delta^2 A_j\right) .
\] (8)

**Numerical examples**

To illustrate the accuracy of calculating the dispersion of plane waves by formula (6), we consider a three-layer periodic medium with anisotropic layers. In the layers, we define orthorhombic anisotropy with horizontal and vertical planes of symmetry. The azimuths of the vertical planes in the three layers of the period, respectively, choose $30^\circ$, $45^\circ$ and $0^\circ$. The densities $\rho$ of the medium in the layers are respectively 2000, 2200, 2100 kg/m$^3$, the thickness of the layers are 3m, 5m, 2m. The vertical velocities of $qP$-waves are respectively 2356, 1568, 1900 m/s, and the vertical velocities of $S_1$-waves are 1225, 603, 850 m/s.

Nonzero elastic coefficients of the three layers of the period are given below ($10^8$N/m$^2$):

<table>
<thead>
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<th>N</th>
<th>$C_{11}$</th>
<th>$C_{12}$</th>
<th>$C_{13}$</th>
<th>$C_{16}$</th>
<th>$C_{22}$</th>
<th>$C_{23}$</th>
<th>$C_{26}$</th>
<th>$C_{33}$</th>
<th>$C_{36}$</th>
<th>$C_{44}$</th>
<th>$C_{45}$</th>
<th>$C_{55}$</th>
<th>$C_{66}$</th>
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<tr>
<td>1</td>
<td>15.387</td>
<td>7.415</td>
<td>6.824</td>
<td>0.323</td>
<td>15.188</td>
<td>6.879</td>
<td>-0.15</td>
<td>11.01</td>
<td>-0.05</td>
<td>3.299</td>
<td>-0.172</td>
<td>3.101</td>
<td>4.21</td>
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<tr>
<td>2</td>
<td>6.141</td>
<td>3.14</td>
<td>2.32</td>
<td>-0.14</td>
<td>6.141</td>
<td>2.32</td>
<td>-0.143</td>
<td>5.409</td>
<td>-0.07</td>
<td>0.9</td>
<td>-0.1</td>
<td>0.9</td>
<td>1.943</td>
</tr>
<tr>
<td>3</td>
<td>10.917</td>
<td>6.636</td>
<td>5.443</td>
<td>0</td>
<td>11.008</td>
<td>3.332</td>
<td>0</td>
<td>7.581</td>
<td>1.96</td>
<td>0</td>
<td>1.517</td>
<td>2.576</td>
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</tr>
</tbody>
</table>

Calculations show that at the boundary between the first pass-band and the first stop-band the plane Floquet waves with different types and horizontal slowness $p_1 = p_2 = 0.1$ s/km have frequencies $F_{qP} = 54.2$ Hz, $F_{s1} = 36.9$ Hz, $F_{s2} = 34.1$ Hz.

It can be seen from Figure1 and Figure2 that the approximation of the increment of the squares of the vertical slowness and phase velocities of the Floquet waves according to formulas (6), (7) is quite accurate in the low-frequency range and satisfactory in the first third of the corresponding pass-band. For example, the relative errors in calculating the dispersion of the phase wave velocities of $qP$, $S_1$, $S_2$ waves for the frequencies $F = 10$ Hz and $F = 20$ Hz are respectively 1.4%, 5.8%, 6.9% and 5.7%, 23%, 27%.

**Conclusions**

Using the second-order perturbation theory and BCH series expansion of the effective system matrix, formulas are derived for calculating the dispersion of the vertical slowness and phase velocities of Floquet waves in a periodic horizontally layered medium with anisotropic layers. These formulas contain a weighted sum of the products of various pairs of increments of the system matrices of the layers relative to the system matrix of the Backus medium and allow a fairly accurate approximate of the dispersion of phase velocities and vertical slowness in the low frequency range. The accuracy of the calculations was demonstrated on a three-layer periodic medium with orthorhombic layers with different azimuths of symmetry planes.

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**References**


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**Figure 1.** Dispersions $\Delta q^2 = q^2(\omega) - q^2(0)$ at $p_1 = p_2 = 0.1$ s / km and their approximations for: (a) $qP$ - wave, (b) $S_1$ - wave, (c) $S_2$ - wave. Solid lines denote exact values, dashed lines denote approximations by the formula (6).

**Figure 2.** Phase velocity dispersion $\Delta v = v(\omega) - v(0)$ at $p_1 = p_2 = 0.1$ s / km and their approximations for: (a) $qP$ - wave, (b) $S_1$ - wave, (c) $S_2$ - wave. Solid lines denote exact values, dashed lines denote approximations by the formula (7).