

GeoTerrace-2020-038**Accuracy of formulas for calculating dynamic heights****E. Novikova, *A. Palamar, I. Yeropunova** (*Kryvyi Rih National University*)**SUMMARY**

In this study, the accuracy of the three known approximate formulae for calculating dynamic heights in the normal gravity field of the Earth for a level ellipsoid of the system WGS-84 is investigated. A simple formula is proposed for the calculation of dynamic heights, the standard deviation of which for geodetic heights varying from 0 to 100 km does not exceed 0.1 m. This formula can be successfully applied not only in geodesy but also in meteorology.

Introduction

In contrast to geodetic heights, defined as the geometric distance from an ellipsoid to a point on the earth's surface, dynamic (geopotential¹) heights do not have a geometric interpretation. However, they are widely used in the design of such engineering structures as roads, airfields, reservoirs (Jekeli, 2001, Savchuk, 2005). But, these heights are most widespread in the study of the Earth's atmosphere and other celestial bodies, as all the characteristics of the atmosphere primarily depend on their location along the level surfaces of the gravity potential (ICAO, 1993, NASA, 2016, Vedel, 2000). It is impossible to design the movement of the aircraft through the atmosphere and even the shape of the aircraft without taking into account the parameters of the atmosphere. Therefore, the software for calculating the shape and aerodynamic parameters of aircraft, created by NASA scientists (Carmichael et al., 1975, Conley, 1992. ANSI/AIAA, 2004), provides the calculation of dynamic altitude. One of the alternative energy sources is windmills. Their design and operation are impossible without taking into account the parameters of the atmosphere, such as wind speed, pressure, air density (Burton et al., 2001). Therefore, the calculation of wind turbine power is accompanied by the determination of dynamic heights. Moreover, studying the dynamics of the atmosphere, climatologists simultaneously study changes in time the level surfaces of the atmosphere in space above the Earth (Kalnay et al., 1996). Thus, a huge amount of material has been accumulated, which allows to analyze the change of potential surfaces not only in space but also in time. In this study, the accuracy of the three most common formulae for calculating the dynamic heights in the normal field of the level ellipsoid WGS-84 was investigated (NIMA, 2000). There is presented the conclusion of another formula for calculating the dynamic height, which in the normal field of the Earth gives minimal error.

Methods of investigation

According to (Hofmann-Wellenhor & Moritz, 2005, Savchuk, 2005), the following formula is used to calculate the dynamic heights:

$$H = \frac{W_0 - W}{\gamma_{45^0}} = \frac{1}{\gamma_{45^0}} \int_0^h g dh \quad (1)$$

The integration is performed along the field line from the initial level surface with a gravity potential W_0 and a height, equal to zero, to a level surface passing through the point P, with a potential W and a height h . In order to calculate the dynamic heights in the normal field of the Earth as accurately as possible, it is necessary to use the first part of formula (1), in which the gravity potential of the normal Earth is determined from equation (Savchuk, 2005): $W = V + U$. (2) where W is the potential of the Earth's gravity, V is the potential of gravitational force, U is the potential of centrifugal force, which are determined by formulae (Moritz, 1992, Savchuk, 2005):

$$V_{norm} = \frac{GM_{norm}}{\rho} \left[1 - \sum_{n=1}^{N_{norm}} J_{2n} \left(\frac{a}{\rho} \right)^{2n} P_{2n,0}(\Phi) \right], \quad U = \frac{\omega^2 \rho^2}{2} \cos^2 \Phi. \quad (3)$$

In formulae (3) ρ, Φ, L are the radius-vector, the geocentric latitude and longitude of the point, located in the gravitational field, respectively, GM is the geocentric gravitational parameter, ω is the angular velocity of rotation of the Earth's gravitational model, J_{2n} are zonal harmonics, $P_{2n,0}(\Phi)$ are corresponding Legendre polynomials. As shown in (Moritz, 2005), to calculate the normal potential of gravity is sufficient to choose $N_{norm} = 12$, since J_{2n} decrease very rapidly. Therefore, the approximation formulae for calculating the dynamic heights were analyzed in the normal field of gravity of the Earth level ellipsoid WGS-84 with $N_{norm} = 12$. To derive approximate formulae for calculating dynamic heights, the second part of formula (1) is used. The first and most popular formula for calculating dynamic heights, based on the fact that according to Newton's law (Hofmann-Wellenhor & Moritz, 2005), the force of attraction is inversely proportional to the square of the distance from the center of the Earth to a point on and above its surface, For two points located on and above the Earth's surface, may be written as:

¹ Traditionally, in climatology, dynamic heights are called geopotential heights (Jekeli, 2001, NASA, 2016).

$$\gamma \approx \gamma_{Earth} \frac{R^2}{(R+h)^2}, \quad (4)$$

where γ_{Earth} is normal force of gravity on the Earth's surface. Since all the results are analyzed in a normal gravity field, h can be interpreted as a geodetic height. Substituting formula (6) in the second part of formula (1) and assuming that γ_{Earth} coincides with the value γ_{45^0} , we obtain the first approximate formula for calculating the dynamic height (Carmichael, 2003):

$$H^{(1)} \approx \int_0^h \frac{R^2}{(R+h)^2} dh = \frac{R}{(R+h)} h. \quad (5)$$

According to (ANSI/AIAA 2004, ICAO, 1993), this formula was used in the construction of models such as U.S. Pat. Standard Atmosphere, 1976 and ICAO standard atmosphere. It should also be noted that this formula is used in software for modeling the shape and trajectory of aircraft motion (Carmichael & Putnam, 1975, Conley, 1992, NASA, 2016), as well as for calculating the power of wind turbines (Burton et al., 2001). Its accuracy was investigated in (Novikova et al., 2017).

A more accurate formula for calculating the dynamic heights can be obtained, if in equation (4) the normal force of gravity on a level ellipsoid γ_{Earth} , is calculated using the Somigliana formula (Moritz, 1992, NIMA, 2000):

$$\gamma_{Earth} \approx \gamma_{Ellipse} = \gamma_0 \left(1 + k \sin^2 \varphi\right) / \sqrt{1 - e^2 \sin^2 \varphi}. \quad (6)$$

where φ is the geodetic latitude, k is the constant of normal gravity, γ_0 is the normal gravity at the equator of the Earth's ellipsoid (NIMA, 2000).

Substituting equations (6, 7) in (1) we obtain a more accurate formula for calculating the dynamic height.

$$H^{(2)} \approx \frac{\gamma_{Ellipse}}{\gamma_{45^0}} \int_0^h \frac{R^2}{(R+h)^2} dh = \frac{\gamma_{Ellipse}}{\gamma_{45^0}} \frac{R}{(R+h)} h. \quad (7)$$

This formula is used in meteorology to calculate dynamic altitudes in modern models of the atmosphere, such as HIRLAM², GRAM³. Traditionally, to calculate the normal force of gravity above the Earth's surface, geodesists use the following formula: $\gamma \approx \gamma_{Ellipse} - \Delta\gamma$, (8) where $\Delta\gamma$ - the free-air reduction. Its approximate value is (Hofmann-Wellenhor & Moritz, 2005): $\Delta\gamma \approx 0.3086 \cdot h$. (9)

In this formula, the coefficient 0.3086 has a dimension. To obtain a reduction in mGal, the height is entered in km. Thus, the dimension of the coefficient in formula (9) is mGal/km. If the height is in meters and the force of gravity - in m/sec², then formula (9) looks like this: $\Delta\gamma \approx 0.3086 \cdot 10^{-8} \cdot h$.

(10) Substituting (8-10) in (1) and performing integration for the case when the force of gravity is determined in m/sec², and height - in meters, we obtain:

$$H^{(3)} \approx \frac{1}{\gamma_{45^0}} \int_0^h (\gamma_{Ellipse} - 0.3086 \cdot 10^{-8} \cdot h) dh = \frac{1}{\gamma_{45^0}} h \left(\gamma_{Ellipse} - \frac{0.3086 \cdot 10^{-8}}{2} h \right). \quad (11)$$

This formula is presented in (Hofmann-Wellenhor & Moritz, 2005) and is used in geodesy and in the design of engineering structures. The modern formula for calculating the free-air reduction according to (NIMA, 2000) has the following form:

$$\Delta\gamma \approx \gamma_{Ellipse} h \left(\frac{2}{a} (1 + f + m) - 2 \sin^2 \varphi - \frac{3}{a^2} h \right). \quad (12)$$

where f is ellipsoidal flattening, $m = \omega^2 a^3 \sqrt{1 - e^2} / GM$.

The numerical values of these quantities for the WGS-84 ellipsoid are presented in (NIMA, 2000).

Using (8, 12), we obtain one more formula for calculating the dynamic height:

² HIRLAM (**H**igh **R**esolution **L**imited **A**rea **M**odel), an international program aimed at developing a numerically accurate short-term weather forecast for Europe. (Vedel, 2000).

³ GRAM (**G**lobal **R**eference **A**tmospheric **M**odel), the atmospheric model created by NASA (ANSI/AIAA, 2004).

$$H^{(4)} \approx \frac{\gamma_{Ellipse}}{\gamma_{45^0}} h \left(1 - \frac{1}{a} \left[(1 + f + m) - 2f \sin^2 \varphi \right] h + \frac{h^2}{a^2} \right). \quad (13)$$

Results of investigations

To assess the accuracy of each of the four formulae (5, 7, 11, 13), the dynamic heights were calculated from a grid with latitudes: $\varphi_0 = 0^0, \varphi_{180} = 90^0, \varphi_i = \varphi_{i-1} + 0,5^0$, and with geodetic heights: $h_0 = 0, h_{1000} = h_{max} = 100\,000\text{ m}, h_j = h_{j-1} + 100\text{ m}$. The standard deviation (SD) of determining the dynamic height at all points of the grid was calculated as follows:

$$\varepsilon^{(K)} = \sqrt{\frac{\sum_{i=0}^{I_{max}} \sum_{j=0}^{J_{max}} (\Delta H_{ij}^{(K)})^2}{(I_{max} \cdot J_{max} - 1)}}, \quad (14) \quad \text{where} \quad \Delta H_{ij}^{(K)} = H_{ij}^{(K)} - H_{ij}, \quad I_{max} = 180,$$

$J_{max} = 1000$. (15) In formula (15) H_{ij} is the value of the dynamic height calculated by formulae (1-3), which can be considered accurate, $H^{(K)}$ is the dynamic height calculated using one of the formulae (5, 7, 11, 13) for $K=1,2,3,4$.

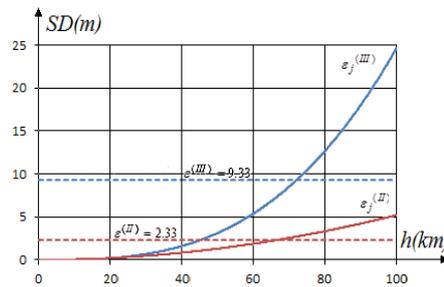


Figure 1 Graph of change SG of calculation the dynamic heights depending on height for (7) and (11) formula

In addition to calculations throughout the all grid, intermediate data were calculated, namely: for the troposphere with $h_{max} = 11\,000\text{ m}, J_{max} = 100$; for points on and near the earth's surface with $h_{max} = 5\,000\text{ m}, J_{max} = 50$. In the table 1 and Fig. 1 present standard deviations and extreme deviations of dynamic heights calculated by approximations and with exact formulae.

Table 1 Accuracy of approximate formulae for calculating dynamic heights

Approximate formula	Interval of geodetic heights								
	from 0 to 100 000 m			from 0 to 11 000 m (troposphere)			from 0 to 5 000 m (Earth surface)		
	SD (m)	$\Delta H_{max}^{(K)}$ (m)	$\Delta H_{min}^{(K)}$ (m)	SD (m)	$\Delta H_{max}^{(K)}$ (m)	$\Delta H_{min}^{(K)}$ (m)	SD (m)	$\Delta H_{max}^{(K)}$ (m)	$\Delta H_{min}^{(K)}$ (m)
(5)	108,98	268,58	-262,44	12,03	29,08	-29,12	5,510	13,204	-13,244
(7)	2,33	8,82	-1,53	0,03	0,11	-0,02	0,006	0,022	-0,004
(11)	9,33	0,00	-25,58	0,01	0,00	-0,05	0,002	0,000	-0,006
(13)	0,11	0,40	-5,6E-08	1,E-04	4,8E-04	-5,6E-08	3,E-05	1,2E-04	-5,6E-08

Conclusions

The largest errors, according to the table. 1 has the formula (5). Even close to the Earth's surface at 5000 m above, it gives an extreme error. Formula (11), used in geodesy and construction, compared to formula (7), used in meteorology, has smaller errors near the Earth's surface. On the contrary, formula (7) demonstrates more accurate values of dynamic heights for points outside the troposphere. However, given that the maximum error in calculating dynamic heights by formula (7) is 9 m, it can be assumed that such an error may affect the accuracy of the weather forecast. Formula (13) proposed for calculating dynamic heights has minimal errors both near the Earth's surface and at points above its surface up to 100 000 m. The maximum calculation error using this formula for an altitude of 100 000 m does not exceed 4 dm. At the upper points of the troposphere, the maximum calculation error using this formula is less than 1 mm. Therefore, formula (13) can be recommended for all variants of

using dynamic heights. The analysis of the formulae is performed for the normal field of gravity of the Earth. For a real field, the errors of all formulae must be even greater and depend not only on the latitude and altitude but also on the longitude of points on and above the Earth's surface.

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