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Construction and accuracy estimation of determining the Earth's geoid by its potential

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SUMMARY

The paper presents an algorithm for constructing a geoid based on the external gravitational field of the Earth, the radius vector of which is determined from the condition of the constancy of the potential on the equipotential surface. The values of the coordinates of such a figure calculated by the iterative method are discrete points in space, and therefore, using them, it is possible to visually construct a three-dimensional geoid image or in the form of contour maps on a plane. A formula for an a priori estimate of the accuracy of determining the radius vector of the Earth's figure is derived, based on the theory of implicit functions of many variables. Approbation of the described technique is carried out on a specific example. The calculation results confirm the convergence of the iterative process in determining the values of the radius vector and a high degree of calculation accuracy (5-6 cm). Therefore, this approach complements traditional assessment methods and can be fully used to study the shape of the Earth.

Introduction

Traditionally, the planet's geoid is determined by calculating its elevation N from the relatively surface (fig.1) using the Bruns formula (Hofmann-Wellenhof Dr. B., Moritz Dr, 2005). According to Gauss, geoid is the "mathematical figure of the Earth", a smooth but irregular surface whose shape results from the uneven distribution of mass within and on the surface of Earth. Despite being an important concept for almost 200 years in the history of geodesy and geophysics, it has been defined to high precision only since advances in satellite geodesy in the late 20th century. Spherical harmonics are often used to approximate the shape of the geoid. The current best such set of spherical harmonic coefficients is EGM96 (Earth Gravity Model 1996), determined in an international collaborative project led by the National Imagery and Mapping Agency (now the National Geospatial-Intelligence Agency, or NGA). EGM96 (Smith, 1998) contains a full set of coefficients to degree and order 360, describing details in the global geoid as small as 55 km. Again, it is possible to calculate geoid directly based on its definition. Indeed, according to the definition of a geoid, it is a surface determined with the equation

$$U\left(x,y,z\right)=U_0 \qquad (1)$$

where U_0 – average potential value on the ocean surface, $U(x,y,z)$ – potential at the point $P\left(x,y,z\right)$ (fig.1). The dependence of one of the coordinates, for example z, on the other two coordinates is supplied by equation (1) in an implicitly form and is established by its solution. Therefore, information about the geoid, in this case, is obtained in digital form or graphically in the form of contour lines or volumetric images. However, the question about the accuracy of determining the radius-vectors, depending on the accuracy of the initial data is arisen.

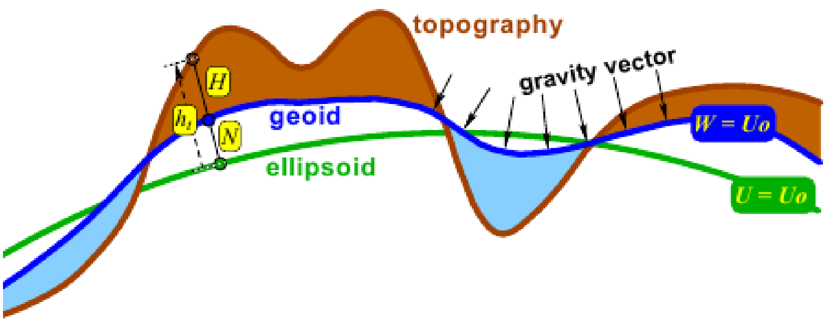

***Figure 1**. Elevation of the geoid relative to the Earth ellipsoid (Barthelmes, 2013)*

Methods

An algorithm for finding radius-vectors in various implementations is presented by a number of researchers. In follows, we use the technique described in (Meshcheryakov, 1991), where the convergence of the iterative process is guaranteed. Traditionally, the potential in some point $\left(r,\vartheta,\lambda\right)$ is presented by series of spherical functions in the spherical coordinate system (Hofmann-Wellenhof Dr. B., et al. 2005):

$$U\left(r,\vartheta,\lambda\right)=\frac{GM}{r}\left(1+\sum_{n=1}^{\infty}\frac{1}{r^{n+1}}\sum_{k=1}^{n}\left(P_n^0\left(\cos\vartheta\right)c_{no}+P_n^k\left(\cos\vartheta\right)\left(c_{nk}\cos k\lambda+s_{nk}\sin k\lambda\right)\right)\right) \qquad (2)$$

where M is a planet's mass; G − gravitational constant; ϑ, λ − geocentric latitude and longitude, respectively; $P_n^k\left(\cos\vartheta\right)$ − associated Legendre polynomials; c_{nk}, s_{nk} − series expansion coefficients. If we also take into account the potential of the rotation force, then we obtain the equality

$$F = \frac{GM}{r}\left(1+\sum_{n=1}^{\infty}\frac{1}{r^{n+1}}\sum_{k=1}^{n}\left(P_n^0\left(\cos\vartheta\right)c_{no}+P_n^k\left(\cos\vartheta\right)\left(c_{nk}\cos k\lambda+s_{nk}\sin k\lambda\right)\right)\right)+ \\ +\frac{\omega^2 r^2}{2}\left(1-P_n^0\left(\cos\vartheta\right)\right)-U_0. \quad (3)$$

Let's bring it to the form:

$$x = K-\sum_{n=2}^{\infty}a_n x^{n+1}-\frac{b}{x^2}, \quad (4)$$

where

$$a_n=\sum_{k=1}^{n}\left(P_n^0\left(\cos\vartheta\right)c_{no}+P_n^k\left(\cos\vartheta\right)\left(c_{nk}\cos k\lambda+s_{nk}\sin k\lambda\right)\right), \\ b=\frac{\omega^2 a_e^2}{2}\left(1-P_n^0\left(\cos\vartheta\right)\right), \\ K=\frac{U_0 a_e}{GM}. \quad (5)$$

In the monograph (Meshcheryakov, 1991) it is proved that the iterative process (4) is convergent for everyone. As a result, we get a geoid in analytic representation. Let us estimate the accuracy of determining the obtained values of radius-vectors describing the surface. For definiteness, we assume that

$$z = f\left(\vartheta,\lambda,c_{20},\ldots,c_{i,j},\ldots,c_{n,n},s_{21},\ldots,s_{i,j},\ldots,s_{n,n}\right). \quad (6)$$

Root mean square error of function (6) are determined (Zazuliak, 2007)

$$m_r^2=\frac{1}{m}\sum_{i=1}^{m}\left(\frac{\partial f}{\partial q_i}\right)^2 m_{q_i}^2, \quad (7)$$

$$q_1,q_2,\ldots,q_m=c_{20},\ldots,c_{i,j},\ldots,c_{n,n},s_{21},\ldots,s_{i,j},\ldots,s_{n,n},\ m=n^2.$$

Partial derivatives of expression (7) are determined from (3) as derivatives of an implicitly given function:

$$\frac{\partial f}{\partial q_i}=-\frac{\partial F}{\partial q_i}:\frac{\partial F}{\partial r}. \quad (8)$$

Therefore, the relation (7) taking into account (8) takes the form

$$m_r^2=\frac{1}{m\left(\frac{\partial F}{\partial r}\right)^2}\sum_{i=1}^{m}\left(\frac{\partial F}{\partial q_i}\right)^2 m_{q_i}^2. \quad (9)$$

It's easy to see that

$$\frac{\partial F}{\partial c_{n0}}=\frac{1}{r^{n+1}}P_n^0\left(\cos\vartheta\right), \\ \frac{\partial F}{\partial c_{nk}}=\frac{1}{r^{n+1}}P_n^k\left(\cos\vartheta\right)\cos k\lambda, \\ \frac{\partial F}{\partial s_{nk}}=\frac{1}{r^{n+1}}P_n^k\left(\cos\vartheta\right)\sin k\lambda, \quad (10)$$

and

$$\frac{\partial F}{\partial r}=-\frac{GM}{r^2}\left(1+\sum_{n=1}^{\infty}\frac{n+1}{r^n}\sum_{k=1}^{n}\left(P_n^0\left(\cos\vartheta\right)c_{no}+P_n^k\left(\cos\vartheta\right)\left(c_{nk}\cos k\lambda+s_{nk}\sin k\lambda\right)\right)\right)+ \qquad (11)$$
$$+\omega^2 r\left(1-P_n^0\left(\cos\vartheta\right)\right).$$

Returning to the previous notation, we get

$$m_r^2=\frac{\left(GM\right)^2}{m\left(\frac{\partial F}{\partial r}\right)^2}\left(\sum_{n=1}^{\infty}\frac{1}{r^{2n+2}}\sum_{k=1}^{n}\left(\left(P_n^0\left(\cos\vartheta\right)\right)^2 m_{c_{no}}^2+\left(P_n^k\left(\cos\vartheta\right)\cos k\lambda\right)^2 m_{c_{nk}}^2+\left(P_n^k\left(\cos\vartheta\right)\sin k\lambda\right)^2 m_{s_{nk}}^2\right)\right). \qquad (12)$$

Purpose

Using the parameters of the Earth's external gravitational field, establish the shape of the Earth's figure and evaluate the accuracy of such a representation.

Results of numerical experiments and discussion of calculation values

To check the effectiveness of the described technique, the following information was taken. First of all, let us discuss the choice of the potential value U_0. Obviously, its value affects the construction of the geoid surface. In this regard, the name "regional geoid" appears. So, taking different values of this quantity for the Baltic and Black Seas gives a difference of the order of 10 cm for heights relative to the reference surface (reference-ellipsoid). This study does not set the task of a detailed study and discussion of this issue, since the main goal of the study is to develop a methodology for assessing the accuracy of determining the radius vectors. Therefore, we took the parameters of the ellipsoid, recommended by GRS-84 (Hofmann, 2005), as well as the parameters of the gravitational field according to the model EGM96 (Lemoine, F.G, 1998, A. Smith, 1998).

Table 1 *GRS-84 parameters*

System	$U_0 * 10^{15} m^3 .\ c^{-2}$	$W_0 * 10^{15} m^3 .\ c^{-2}$	$a_1 * 10^7 .m$	$w * 10^{-4} .\ rad\ .\ c^{-1}$	a
GRS-84	0.39860050±0.05	0.6263686085	0.6378138	0.7292115147	298.257

Table 2 *Values of geoid radius-vectors according to Stokes constants of the twentieth order due to the EGM96 model*

	$r=\left(r-6300000\right)$, m						
	0^0	60^0	120^0	180^0	240^0	300^0	360^0
0^0	56769.804	56769.804	56769.804	56769.804	56769.804	56769.804	56769.804
30^0	59351.214	59347.588	59344.416	59343.419	59343.969	59348.564	59351.214
60^0	64530.601	64526.306	64527.061	64527.894	64523.927	64527.256	64530.601
90^0	67134.060	67130.348	67136.602	67141.257	67134.925	67133.819	67134.060
120^0	64542.387	64538.551	64539.745	64541.475	64538.533	64540.568	64542.387
150^0	59344.636	59341.869	59336.712	59333.473	59334.257	59340.255	59344.636
180^0	56726.931	56726.931	56726.931	56726.931	56726.931	56726.931	56726.931

Table 3 *Errors in calculating the values of the geoid radius-vectors on the Stokes constants up to the twentieth order according to the EGM96 model*

	The error in the values of radius vectors, m_r						
	0^0	60^0	120^0	180^0	240^0	300^0	360^0
0^0	0.058556	0.058556	0.058556	0.058556	0.058556	0.058556	0.058556
30^0	0.012283	0.012282	0.012283	0.012283	0.012283	0.012282	0.012283

60^0	0.008643	0.008643	0.008643	0.008643	0.008643	0.008643	0.008643
90^0	0.008683	0.008683	0.008683	0.008683	0.008683	0.008683	0.008683
120^0	0.008643	0.008643	0.008643	0.008643	0.008643	0.008643	0.008643
150^0	0.012283	0.012283	0.012283	0.012283	0.012283	0.012283	0.012283
180^0	0.058563	0.058563	0.058563	0.058563	0.058563	0.058563	0.058563

Recommendations and conclusions

Thus, along with obtaining the numerical values of the radius vectors, it becomes possible to obtain the accuracy of their calculation, given in table 3. Based on the numerical values, it is possible to construct both the geoid and the errors in its determination in two versions: in a volumetric image and using contour maps. The values given in the table are given in numerical format, which allows assessing the degree of reliability in determining the radius-vectors of the geoid. Analysis of the numerical data in the table 3 shows that the error is practically independent of longitude, and in latitudinal projection its maximum is reached for the values $\varphi = 30^\circ$ and $\varphi = 150^\circ$. In addition, the maximum value of the error in determining the radius-vectors of the geoid surface by expression (1) is 5-6 centimeters.

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