

21103**Fundamental singularity points in elliptical orthorhombic media**

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SUMMARY

The fundamental S waves singularity points in elliptical orthorhombic media are defined. It is shown that each elliptical orthorhombic model contains only one singularity point located in one of the symmetry planes depending on the magnitude of S wave stiffness coefficients. The S wave slowness surfaces are also defined.

Introduction

An elliptical anisotropy model is very important in seismic anisotropy application since it provides the background (or reference) medium later to be perturbed by anelliptic parameters. Seismic waves have almost elliptical wavefronts (Helbig, 1983), and, therefore, it is important to analyze the properties of all wave modes in the case of elliptical anisotropy. For an orthorhombic model, it is often preferred to refer to this model as “ellipsoidal” but we would prefer to keep reference “elliptical”. There are various applications for an elliptical model (Levin, 1978; Golikov and Stovas, 2012). This approach is successfully used for VTI (transverse isotropic model with a vertical symmetry axis) model (Fomel and Stovas, 2010; Stovas, 2010; Alkhalifah, 2011; Stovas and Fomel, 2012; Sripanich and Fomel, 2015), TTI (transverse isotropic model with a tilted symmetry axis) model (Stovas and Alkhalifah, 2012; 2013) and ORT (orthorhombic) and TOR (tilted orthorhombic) models (Sripanich and Fomel, 2015; Stovas, 2015; Hao and Stovas, 2016; Hao et al., 2016; Stovas et al., 2016; Sripanich et al, 2017; Xu et al., 2017; 2018; Stovas and Fomel, 2019). Elliptically anisotropic models are successfully used for wave extrapolation methods (Waheed and Alkhalifah, 2015) and for attenuating anisotropic media (Hao and Alkhalifah, 2017). Elliptical models are typically simpler compared to its anelliptic counterpart, yielding simpler and more efficient equations. In most cases, the elliptical model is defined for P waves only since the S waves elliptical model is considered to be too complex. There are many numerical approaches to specify an elliptical model of different complexity in anisotropy (Burridge et al., 1993; Ettrich et al., 2001).

In this paper, we define P, S1 and S2 waves in an elliptical orthorhombic medium. We show that all wave modes in such a medium are generally coupled with an isolated coupling term. When the coupling term is neglected, the P, S1 and S2 waves are referred to as the fundamental waves in an elliptical orthorhombic medium. The fundamental S-waves in an elliptical ORT model produce one singularity point and the out-of-plane triplication (Stovas et al., 2021) associated with this point.

Elliptical orthorhombic media

The elliptical orthorhombic medium is defined by applying the idea similar to what is used for transversely isotropic medium above for all symmetry planes. The conditions for all off-diagonal stiffness coefficients can be written as follows: $c_{12} = \sqrt{(c_{11} - c_{66})(c_{22} - c_{66})} - c_{66}$, $c_{13} = \sqrt{(c_{11} - c_{55})(c_{33} - c_{55})} - c_{55}$ and $c_{23} = \sqrt{(c_{22} - c_{44})(c_{33} - c_{44})} - c_{44}$. This implies that in case of an elliptical ORT model, there are only six independent (on-diagonal) stiffness coefficients. We select the standard ORT model from Schoenberg and Helbig (1997) with off-diagonal stiffness coefficients being computed as shown above. The diagonal stiffness coefficients from the standard ORT model are $c_{11} = 9$ GPa, $c_{22} = 9.84$ GPa, $c_{33} = 5.9375$ GPa, $c_{44} = 2$ GPa, $c_{55} = 1.6$ GPa, and

$c_{66} = 2.182$ GPa. In an elliptical ORT medium, the Christoffel matrix $\mathbf{G} = (g_{ij})$, $i, j = 1, 2, 3$, and

$$g_{ji} = g_{ij} : g_{11} = c_{11}n_1^2 + c_{66}n_2^2 + c_{55}n_3^2, g_{22} = c_{66}n_1^2 + c_{22}n_2^2 + c_{44}n_3^2, g_{33} = c_{55}n_1^2 + c_{44}n_2^2 + c_{33}n_3^2,$$

$$g_{12} = \sqrt{(c_{11} - c_{66})(c_{22} - c_{66})}n_1n_2, g_{13} = \sqrt{(c_{11} - c_{55})(c_{33} - c_{55})}n_1n_3, g_{23} = \sqrt{(c_{22} - c_{44})(c_{33} - c_{44})}n_2n_3$$

where $\mathbf{n} = (n_1, n_2, n_3)$ is a unit directional vector in the phase domain, and c_{ij} are the stiffness

coefficients. The phase velocities squared v^2 for P-, S1- and S2-waves are given by the eigenvalues of Christoffel matrix with elements defined in equation 6, $\det(\mathbf{G} - v^2\mathbf{I}) = 0$, where \mathbf{I} is an identity 3x3 matrix. The characteristic (Christoffel) equation for an elliptical ORT medium can be written in the following form,

$$F_{ORT} = (v^2 - v_{Pe}^2)(v^4 - rv^2 + s) + f = 0, \tag{1}$$

where the P-wave term is defined by well-known phase velocity ellipsoid (the fundamental approximation for P waves), $v_{Pe}^2 = c_{11}n_1^2 + c_{22}n_2^2 + c_{33}n_3^2$, and S-wave term is defined by coefficients

$$r = c_{44}(n_2^2 + n_3^2) + c_{55}(n_1^2 + n_3^2) + c_{66}(n_1^2 + n_2^2), s = c_{44}c_{55}n_3^2 + c_{44}c_{66}n_2^2 + c_{55}c_{66}n_1^2. \tag{2}$$

The coupling term f is given by the relative directional scaling of P- and S-wave ellipsoids,

$$f = \left(\sqrt{(c_{11} - c_{55})(c_{22} - c_{66})(c_{33} - c_{44})} - \sqrt{(c_{11} - c_{66})(c_{22} - c_{44})(c_{33} - c_{55})} \right)^2 n_1^2 n_2^2 n_3^2. \quad (3)$$

The phase unit vector is specified with projections $n_1 = \sin \theta \cos \phi$, $n_2 = \sin \theta \sin \phi$ and $n_3 = \cos \theta$, where θ and ϕ are the phase polar and phase azimuth angles, respectively. We can see from equation 6 that all waves are coupled due to the presence of coupling term f . Let us analyze the coupling term.

One can see that $f \geq 0$, and f_{\max} is achieved when $n_1 = n_2 = n_3 = 1/\sqrt{3}$.

The decoupling conditions ($f = 0$) are the following:

(a) Symmetry planes, $n_j = 0$, $j = 1, 2, 3$. The coupling term is zero in all symmetry planes. The P-wave phase velocity is given by an elliptical equation. For example, if $n_2 = 0$, ((n_1, n_3) being a symmetry plane) equation 9 reduces to $v_{Pe}^2 = c_{11}n_1^2 + c_{33}n_3^2$, and the S-wave equation takes the TI-form (S waves are decoupled into SV and SH waves), $v^4 - rv^2 + s = (v^2 - c_{55})(v^2 - c_{66}n_1^2 - c_{44}n_3^2) = 0$.

(b) Isotropic P-wave, $c_{11} = c_{22} = c_{33} = c_p$. In this case, we have, $v_{Pe}^2 = c_p$, and we end up with coupled S-waves.

(c) Isotropic S wave, $c_{44} = c_{55} = c_{66} = c_s$. In this case, the P-wave phase velocity is given by equation 9, and S-wave phase velocity reduces to a circle, $v_{S1e}^2 = v_{S2e}^2 = c_s$.

The special case of acoustic elliptical medium is defined by $c_s = 0$. This case shows that S1 and S2 wave artifacts in acoustic orthorhombic medium are fully controlled by the anelliptic parameters.

(d) One of the symmetry planes is isotropic. Symmetry plane (n_1, n_2) is isotropic (VTI case), $c_{44} = c_{55}$ and $c_{22} = c_{11}$. Symmetry plane (n_2, n_3) is isotropic (HTI case), $c_{66} = c_{55}$ and $c_{22} = c_{33}$. Symmetry plane (n_1, n_3) is isotropic (also an orthogonal HTI case), $c_{66} = c_{44}$ and $c_{11} = c_{33}$. In this case, all waves are decoupled.

The coupling term is computed for the standard ORT model (Schoenberg and Helbig, 1997) and plotted in Figure 1. We can see that the magnitude of f is very small compared with any wave mode phase velocities to the sixth power. The maximum value $f_{\max} \approx 0.0031 \text{ km}^6/\text{s}^6$.

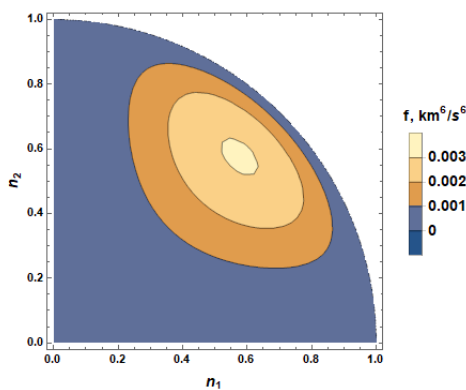


Figure 1 The coupling term f computed for the standard ORT model.

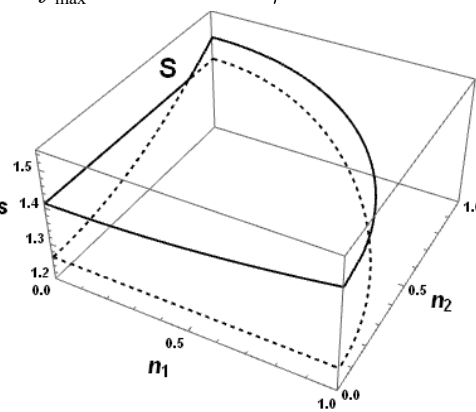


Figure 2 The S1-wave (solid line) and S2-wave (dashed line) phase velocities for the standard ORT model in symmetry planes. The singularity point in symmetry plane (n_2, n_3) is annotated as **S**.

Let us start with the decoupled ($f = 0$) S-wave equation $v^4 - rv^2 + s = 0$ with coefficients defined in equation (2). The condition for singularity points can be obtained by setting the discriminant of this equation to zero, i.e. $D_0 = r^2 - 4s$. If $c_{44} = c_{55}$, the singularity point is located on the vertical symmetry axis $(0, 0, 1)$; if $c_{44} = c_{66}$ or $c_{55} = c_{66}$, the singularity point is located on the horizontal symmetry axis

$(0,1,0)$ and $(1,0,0)$, respectively. The non-trivial solution of equation $D_0 = 0$ gives only one root in one of the symmetry planes. The selection of this symmetry plane is dependent on the magnitude of the S-wave stiffness coefficients as follows:

$$\begin{aligned} \min(c_{44}, c_{66}) \leq c_{55} \leq \max(c_{44}, c_{66}), \quad & \text{symmetry plane } (n_1, n_3), \\ \min(c_{55}, c_{66}) \leq c_{44} \leq \max(c_{55}, c_{66}), \quad & \text{symmetry plane } (n_2, n_3), \\ \min(c_{44}, c_{55}) \leq c_{66} \leq \max(c_{44}, c_{55}), \quad & \text{symmetry plane } (n_1, n_2). \end{aligned} \quad (4)$$

Since the coupling term $f = 0$ in the symmetry planes, therefore, it does not affect the singularity point mentioned above. There are no singularity points outside the symmetry planes. The behavior of the S-wave phase velocities in symmetry planes is illustrated in Figure 2. For selected model, the singularity point is in symmetry plane (n_2, n_3) . The singularity point defined above is a fundamental singularity point in an ORT model. All other singularity points (Ivanov and Stovas, 2017) in an anelliptic ORT model are induced by anelliptic parameters. We also compute the group velocity surfaces for S1 and S2 waves (Figure 3). We can clearly see the S2-wave out-of-plane triplication with a complex geometry. The P-wave slowness surface computed from equation (3) is trivial, $c_{11}p_1^2 + c_{22}p_2^2 + c_{33}p_3^2 = 1$,

where p_1, p_2, p_3 are projections of the slowness vector. The S-wave slowness surface is computed from the S-wave term in equation (3) and given by the fourth-order equation

$$\begin{aligned} c_{44}c_{55}p_3^4 - \left[c_{44}(1 - c_{55}(p_1^2 + p_2^2) - c_{55}p_2^2) + c_{55}(1 - c_{66}p_1^2) \right] p_3^2 \\ + (1 - c_{66}(p_1^2 + p_2^2))(1 - c_{55}p_1^2 - c_{44}p_2^2) = 0. \end{aligned} \quad (5)$$

The singularity point for the standard ORT model (Schoenberg and Helbig, 1997) is located in (p_2, p_3) symmetry plane with $p_2^2 = (c_{44} - c_{55})/c_{44}(c_{66} - c_{55})$, $p_3^2 = (c_{66} - c_{55})/c_{44}(c_{44} - c_{55})$.

Conclusions

We show that there exists a coupling between the P and S waves in an elliptical orthorhombic medium. We also derive equations for the fundamental P and S waves in elliptical orthorhombic media when the coupling term is neglected. The S-wave equation supports a fundamental singularity point in one of the symmetry planes. The non-zero coupling term does not bring any additional singularity points outside of the symmetry planes. We show that this singularity point results in an out-of-plane triplication for S2 waves. Analysis of the anellipticity for P and S waves shows that the coupling term contributes in cross-term anellipticity for all wave modes. In addition to that, the S-wave anellipticities have an additional term defined by differences in the S-wave stiffness coefficients.

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